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Pattern formation induced by additive noise: a moment-based analysis

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Abstract. We analyze the condition for instability and pattern formation induced by additive noise in spatially extended systems. The approach is based on consideration of higher moments which extract out nonlinearities of appropriate order. Our analysis reveals that cubic nonlinearity plays a crucial role for the additive noise to a leading order that determines the instability threshold which is corroborated by numerical simulation in two specific reaction-diffusion systems.

PACS. 82.40.Ck Pattern formation in reactions with diffusion, flow and heat transfer – 47.54.+r Pattern selection; pattern formation – 05.45.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 47.20.Ky Nonlinearity (including bifurcation theory)

1 Introduction

The effect of noise on zero-dimensional and spatially extended systems has been the subject of wide interest over the last two decades. Various noise-induced phenomena have been discovered in the systems far from equilibrium with the demonstration that noise can stabilize and reorganize a system which can exhibit counterintuitive dynamical behaviour. The wellknown recent examples are noise-induced transitions [1,2], stochastic resonance[3,4], noise-induced phase transition [5–10], noise-induced spatiotemporal intermittency [11], noise-induced front propagation of traveling waves [12] etc. Of these, noise-induced pattern formation [13,14,16–18] plays a key role in understanding the problem of instability in several issues which include for example, Rayleigh-Benard convection in fluids described by Swift-Hohenberg equation [19], self-replication in biological systems described by Gray-Scott model [20] etc. Noise in both additive and multiplicative forms are in use in all these studies. For multiplicative noise processes one may proceed to carry out a linear analysis of stability with the help of Novikov's theorem so that it is possible to figure out the contribution of the noise properties in calculating the stability and bifurcation boundaries [5,14]. Although no such method is directly applicable when the noise is additive, a number of methods have been advocated over the years for exploring such situations. The wellknown mean field method [5,8] takes care of evolution of distribution function of relevant variables in terms of Fokker-Plank equation and the associated Langevin equation to calculate the mean value of the variables self consistently. Although the procedure traditionally neglects higher order correlations, an interesting approach based on cumulant dynamics has recently been put forward to predict noise induced phase transition [15] which includes the effect of higher order correlations. The linearized treatment of reaction-diffusion systems with additive noise in terms of structure function [5] has also met considerable success in explaining noise-induced phase transition and related issues. Several techniques for the treatment of noise-induced effect within stochastic linearization schemes [21,22] have received attention particularly for zero dimensional systems. It has been pointed out that the method based on moments can be used to solve a class of stochastic optimization problems. These studies, notwithstanding, the onset of instability induced by additive noise deserves special attention so far as the specific role of nonlinearity in pattern formation in spatially extended systems is concerned.

The object of the present paper is to develop a method based on higher order moments for reaction-diffusion systems driven by additive noise for determination of noise-induced instability threshold for pattern formation. As we have already pointed out that the calculation of linear moments using Novikov's theorem is not sufficient to capture the effect of additive noise as in the case of multiplicative noiser, the consideration of higher order moments without factorization to lower moments is imperative for the treatment of interplay of nonlinearity and stochasticity. In what follows we construct a set of linear equations of higher nth order moments using truncation at $(n+1)$th moment to close the system of equations. Our analysis reveals that the cubic nonlinearity plays the leading order role in extracting out the effect of noise through the higher

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order moments. Our theoretical analysis is corroborated by numerical simulations on two model reaction-diffusion systems.

2 Analysis of higher order moments and the condition for noise-induced instability

To analyze the effect of additive noise on a homogeneous steady state to induce instability, we consider a general two-component reaction-diffusion system in two dimensions described by the following set of equations

$$u_t = f(u,v) + u_{xx} + u_{yy} \quad (1)$$

$$v_t = g(u,v) + dv_{xx} + dv_{yy} \quad (2)$$

$u(x,y,t)$ and $v(x,y,t)$ being the dimensionless concentrations of the activator and inhibitor, respectively and $f(u,v)$ and $g(u,v)$ describe the kinetics of the reaction; d refers to the ratio of the diffusion coefficients of the two species.

Let $u = u_0$ and $v = v_0$ be the homogeneous steady state of the system such that
$f(u_0,v_0) = g(u_0,v_0) = 0$. We assume further that this state is linearly stable in absence of diffusion, i.e., $f_u+f_v < 0$ and $f_u g_v - g_u f_v > 0$, where the derivatives of f and g with respect to u or v are evaluated at the steady state.

We now consider a small spatio-temporal perturbation $\delta u(x,y,t)$ and $\delta v(x,y,t)$ on this steady state u_0, v_0 as $u = u_0+\delta u$ and $v = v_0+\delta v$ and by expanding the reaction terms around this state in a Taylor series up to third order we obtain,

$$\begin{aligned}\delta u_t &= f_u\delta u + f_v.\delta v + (1/2)f_{uu}\delta u^2 \\ &\quad + (1/2)f_{vv}.\delta v^2 + f_{uv}\delta u.\delta v + (1/6)f_{uuu}.\delta u^3 \\ &\quad + (1/6)f_{vvv}.\delta v^3 + (1/2)f_{uuv}\delta u^2.\delta v \\ &\quad + (1/2)f_{uvv}\delta u\delta v^2 + \delta u_{xx} + \delta u_{yy} \end{aligned} \quad (3)$$

$$\begin{aligned}\delta v_t &= g_u\delta u + g_v\delta v + (1/2)g_{uu}\delta u^2 + (1/2)g_{vv}\delta v^2 \\ &\quad + g_{uv}\delta u\delta v + (1/6)g_{uuu}\delta u^3 + (1/6)g_{vvv}\delta v^3 \\ &\quad + (1/2)g_{uuv}\delta u^2\delta v + (1/2)g_{uvv}\delta u\delta v^2 \\ &\quad + d(\delta v_{xx} + \delta v_{yy}). \end{aligned} \quad (4)$$

The system is now subjected to an external additive noise $\eta(x,y,t)$ on u and another $\xi(x,y,t)$ on v, both being white and Gaussian in nature and random in space and time. Their averages and correlations are given by

$$\langle\eta(x,y,t)\rangle = \langle\xi(x,y,t)\rangle = 0 \quad (5)$$

$$\langle\eta(x,y,t)\eta(x',y',t')\rangle = 2D_u.\delta(x-x')\delta(y-y')\delta(t-t') \quad (6)$$

$$\langle\xi(x,y,t)\xi(x',y',t')\rangle = 2D_v.\delta(x-x')\delta(y-y')\delta(t-t'). \quad (7)$$

D_u and D_v are the noise strengths for the additive noises corresponding to u and v variables respectively. The cross correlation between the noises is neglected. With these additional noise terms the system of equations now look like

$$\begin{aligned}\delta u_t &= f_u\delta u + f_v\delta v + (1/2)f_{uu}\delta u^2 + (1/2)f_{vv}\delta v^2 \\ &\quad + f_{uv}\delta u\delta v + (1/6)f_{uuu}\delta u^3 + (1/6)f_{vvv}\delta v^3 \\ &\quad + (1/2)f_{uuv}\delta u^2\delta v + (1/2)f_{uvv}\delta u\delta v^2 \\ &\quad + \delta u_{xx} + \delta u_{yy} + \eta(x,y,t) \end{aligned} \quad (8)$$

$$\begin{aligned}\delta v_t &= g_u\delta u + g_v\delta v + (1/2)g_{uu}\delta u^2 + (1/2)g_{vv}\delta v^2 \\ &\quad + g_{uv}\delta u\delta v + (1/6)g_{uuu}\delta u^3 + (1/6)g_{vvv}.\delta v^3 \\ &\quad + (1/2)g_{uuv}\delta u^2\delta v + (1/2)g_{uvv}\delta u\delta v^2 \\ &\quad + d(\delta v_{xx} + \delta v_{yy}) + \xi(x,y,t). \end{aligned} \quad (9)$$

Expressing the spatiotemporal perturbations $\delta u(x,y,t)$, $\delta v(x,y,t)$ and the noise perturbation $\eta(x,y,t)$ and $\xi(x,y,t)$ in the form

$$\delta u(x,y,t) = U(x,y,t)\cos k_x x \cos k_y y \quad (10)$$

$$\delta v(x,y,t) = V(x,y,t)\cos k_x x \cos k_y y \quad (11)$$

$$\eta(x,y,t) = \bar{\eta}(x,y,t)\cos k_x x \cos k_y y \quad (12)$$

$$\xi(x,y,t) = \bar{\xi}(x,y,t)\cos k_x x \cos k_y y \quad (13)$$

and upon inserting them in equations (8),(9) we collect the terms containing $\cos k_x x \cos k_y y$ of both sides of the resulting equations. Before proceeding further three points are to be emphasized here. First we are interested in amplitudes $U(x,y,t)$ and $V(x,y,t)$ of the spatiotemporal perturbations corresponding to fundamental wave vector components k_x, k_y only, which can be selected out by the suitable choice of length of the reaction domain L_x and L_y along x and y directions. We may therefore neglect the components corresponding to higher spatial harmonics $\cos 2k_x x$, $\cos 2k_y y$, $\cos 3k_x x$, $\cos 3k_y y$ and so on wherever they arise in dealing with nonlinear terms. For example, the cubic anharmonicity results in terms like $\cos^3 k_x x$ which can be expressed as $\cos^3 k_x x = 1/4.(\cos 3k_x x + 3\cos k_x x)$. We keep the $\cos k_x x$ term neglecting the third harmonic term $\cos 3k_x x$. Secondly, it is also expected that since $\eta(x,y,t)$ and $\xi(x,y,t)$ are external noise perturbations, the amplitudes of these perturbations corresponding to the fundamental wave vectors k_x, k_y are expected to affect the system in most significant way. This makes $\bar{\eta}$ and $\bar{\xi}$ precisely relevant for the noise induced effects we search for and gives the motivation for choosing the form of $\eta(x,y,t)$ and $\xi(x,y,t)$ as given in (12) and (13). Third, it is easy to note that insertion of equations (10–13) in equations (8) and (9) yields terms $\bigtriangledown^2 U$ and $\bigtriangledown^2 V$ containing $\cos k_x x \cos k_y y$. Since for initiation of an instability leading to pattern we look for a sizable number of nodes along x and y directions so that the system size can accommodate a few wavelengths, it is convenient to assume $k^2 U \gg \bigtriangledown^2 U$ and $k^2 V \gg \bigtriangledown^2 V$. This implies that the spatial variation of the envelope functions U and V are much smaller compared to the wavelengths ($k \sim 1/\lambda$) or in other words we assume a slowly varying envelope approximation which has been widely used in optics and other areas.

Based on these considerations we are now led to the following equations for the envelope functions $U(x,y,t)$ and $V(x,y,t)$,

$$U_t = f_u U + f_v V + (3/32) f_{uuu} U^3 + (3/32) f_{vvv} V^3 + (9/32) f_{uuv} U^2 V + (9/32) f_{uvv} U V^2 - k^2 U + \overline{\eta} \quad (14)$$

$$V_t = g_u U + g_v V + (3/32) g_{uuu} U^3 + (3/32) g_{vvv} V^3 + (9/32) g_{uuv} U^2 V + (9/32) g_{uvv} U V^2 - dk^2 V + \overline{\xi} \quad (15)$$

where

$$k^2 = k_x^2 + k_y^2. \quad (16)$$

At this point it is important to define the noise characteristics of $\bar{\eta}(x,y,t)$ and $\bar{\xi}(x,y,t)$ in relation to $\eta(x,y,t)$ and $\xi(x,y,t)$. To this end we assume

$$\langle \overline{\eta}(x,y,t) \rangle = \langle \overline{\xi}(x,y,t) \rangle = 0 \quad (17)$$

$$\langle \overline{\eta}(x,y,t) \overline{\eta}(x',y',t') \rangle = 2.\boldsymbol{D_u}.\delta(x-x')\delta(y-y')\delta(t-t') \quad (18)$$

$$\langle \overline{\xi}(x,y,t) \overline{\xi}(x',y',t') \rangle = 2.\bar{D}_v.\delta(x-x')\delta(y-y')\delta(t-t'). \quad (19)$$

The noise strengths $\bar{D}_u$ and $\bar{D}_v$ are to be determined as follows: Putting equation (12) in equation (6) we obtain

$$\langle \eta(x,y,t)\eta(x',y',t') \rangle = \langle \overline{\eta}(x,y,t)\overline{\eta}(x',y',t') \rangle \cos k_x x \cos k_x x' \cos k_y y \cos k_y y'. \quad (20)$$

Expressing $x' = x + \varDelta$, $y' = y + \epsilon$ and $t' = t + \tau$ in equation (20) yields on integration, the following relation

$$\int \langle \eta(x,y,t)\eta(x+\varDelta, y+\epsilon, t+\tau) \rangle d\tau d\varDelta d\epsilon = \int \langle \bar{\eta}(x,y,t)\bar{\eta}(x+\varDelta, y+\epsilon, t+\tau) \rangle \times \cos k_x x \cos k_x (x+\varDelta) \cos k_y y \cos k_y (y+\epsilon) \times d\tau d\varDelta d\epsilon. \quad (21)$$

Making use of relation (18) in (21) results in right hand side of (21)

$$2\bar{D}_u \int \delta(\varDelta)\delta(\epsilon)\delta\tau \times \cos k_x x \cos k_x (x+\varDelta) \times \cos k_y y \cos k_y (y+\epsilon) = 2\bar{D}_u \cos^2 k_x x \cos^2 k_y y. \quad (22)$$

The spatial averaging of (22) over x and y gives

$$\left\langle \int \langle \eta(x,y,t)\eta(x+\varDelta, y+\epsilon, t+\tau) \rangle d\tau d\varDelta d\epsilon \right\rangle = 2\bar{D}_u (1/4). \quad (23)$$

On the other hand, equation (6) on similar integration and spatial averaging yields

$$\left\langle \int \langle \eta(x,y,t)\eta(x+\varDelta, y+\epsilon, t+\tau) d\tau d\varDelta d\epsilon \right\rangle = 2D_u. \quad (24)$$

From equations (23) and (24) we obtain the relation

$$\bar{D}_u = 4D_u. \quad (25)$$

Proceeding exactly in the same way it may be shown

$$\bar{D}_v = 4D_v \quad (26)$$

for the noise ξ and $\bar{\xi}$. The relation (25) and (26) therefore completely defines the noise properties (18) and (19)

It is worth-noting here that the third order terms contribute to equations (14, 15) in a significant way. The heart of the present analysis lies in exploring the essential role of these leading order nonlinear terms in generating instability induced by additive noise.

In a discrete lattice of N-cells it is convenient to write the stochastic differential equations (14) and (15) in the following form (discretization implies the space point $(x, y) \rightarrow (i, j)$-lattice site)

$$\delta U_{ij}(t)/\delta t = f_u U_{ij} + f_v V_{ij} + (3/32) f_{uuu} U_{ij}^3 + (3/32) f_{vvv}.V_{ij}^3 + (9/32) f_{uuv} U_{ij}^2 V_{ij} + (9/32) f_{uvv} U_{ij} V_{ij}^2 - k^2 U_{ij} + \overline{\eta_{ij}} \quad (27)$$

$$\delta V_{ij}(t)/\delta t = g_u U_{ij} + g_v.V_{ij} + (3/32) g_{uuu} U_{ij}^3 + (3/32) g_{vvv}.V_{ij}^3 + (9/32) g_{uuv} U_{ij}^2 V_{ij} + (9/32) g_{uvv} U_{ij} V_{ij}^2 - dk^2 V_{ij} + \overline{\xi_{ij}}. \quad (28)$$

In this discretized space the noise correlations acquire the form:

$$\langle \overline{\eta_{ij}}(t) \overline{\eta_{kl}}(t') \rangle = 2C^u \delta_{ik}\delta_{jl}\delta(t-t') \quad (29)$$

$$\langle \overline{\xi_{ij}}(t) \overline{\xi_{kl}}(t') \rangle = 2C^v \delta_{ik}\delta_{jl}\delta(t-t') \quad (30)$$

where $C^u = 4.D_u/\triangle x.\triangle y$ and $C^v = 4.D_v/\triangle x.\triangle y$ for the two-dimensional reaction domain.

Equations (27) and (28) together with the noise correlation functions given by (29) and (30) serve as the starting point for generating the equations for the higher moments. To this end we proceed by constructing the equations for the first moments $\langle U_{ij}(t) \rangle$ and $\langle V_{ij}(t) \rangle$ from the equations (27) and (28). For the sake of brevity we drop the subscripts ij from variables U_{ij} and V_{ij} and write simply

$$\begin{aligned}\langle U_t\rangle &= f_u\langle U\rangle + f_v\langle V\rangle + (3/32)f_{uuu}\langle U^3\rangle \\ &+(3/32)f_{vvv}\langle V^3\rangle + (9/32)f_{uuv}\langle U^2V\rangle \\ &+(9/32)f_{uvv}\langle UV^2\rangle - k^2\langle U\rangle \end{aligned} \quad (31)$$

$$\begin{aligned}\langle V_t\rangle &= g_u\langle U\rangle + g_v\langle V\rangle + (3/32)g_{uuu}\langle U^3\rangle \\ &+(3/32)g_{vvv}\langle V^3\rangle + (9/32)g_{uuv}\langle U^2V\rangle \\ &+(9/32)g_{uvv}\langle UV^2\rangle - dk^2\langle V\rangle. \end{aligned} \quad (32)$$

It is immediately apparent that equations (31) and (32) for the first moments $\langle U\rangle$ and $\langle V\rangle$ require the knowledge of higher moments. To construct them we may proceed with the following example. For calculation of $\langle \dot{U^2}\rangle$ we first note that

$$d\langle U^2\rangle/dt = 2\langle U\dot{U}\rangle. \quad (33)$$

To construct the equations of motion for second moment $\langle U^2\rangle$ we multiply equation (27) by 2U to give

$$\begin{aligned}dU^2/dt &= 2f_uU^2 + 2f_uUV + (3/16)f_{uuu}U^4 \\ &+(3/16)f_{vvv}UV^3 + (9/16)f_{uuv}.U^3V \\ &+(9/16)f_{uvv}U^2V^2 - 2k^2U^2 + 2\bar{\eta}U. \end{aligned} \quad (34)$$

Thus although the parent equations (27) and (28) are stochastic equations with additive noise, equation (34) is an equation with multiplicative noise because of the occurrence of the last term. We then make use of Novikov's theorem for averaging in discretized space for the Gaussian noise process i.e.,

$$\langle F(U_{ij})\bar{\eta}_{ij}\rangle = C^u\langle F(U_{ij})F'(U_{ij})\rangle \quad (35)$$

$$\langle F(V_{ij})\bar{\eta}_{ij}\rangle = C^v\langle F(V_{ij})F'(V_{ij})\rangle \quad (36)$$

where F is the function of U or V and F' is the derivative with respect to U or V. Therefore we have on averaging over (34) with the help of equation (35), the equation of the moment $\langle U^2\rangle$

$$\begin{aligned}\langle U^2\rangle &= 2f_u\langle U^2\rangle + 2f_v\langle UV\rangle + (3/16)f_{uuu}\langle U^4\rangle \\ &+(3/16)f_{vvv}\langle UV^3\rangle + (9/16)f_{uuv}\langle U^3V\rangle \\ &+(9/16)f_{uvv}\langle U^2V^2\rangle - 2k^2\langle U^2\rangle + 2C^u\langle U\rangle. \end{aligned} \quad (37)$$

Proceeding in the same way we make use of the equations (27) and (28) and (35), (36) for averaging over noise to construct the equations for the moments $\langle UV\rangle$, $\langle V^2\rangle$, $\langle U^3\rangle$, $\langle V^3\rangle$, $\langle U^2V\rangle$, $\langle UV^2\rangle$. It is important to note that the equations thus constructed form an infinite hierarchy. One therefore must truncate the hierarchy at a particular moment. In order to capture the leading order effects of additive noise we now take care of the moments upto third order, or in other words we neglect the contribution of the terms from $\langle UV^3\rangle$, $\langle U^3V\rangle$, $\langle U^4\rangle$, $\langle V^4\rangle$ etc. and beyond while constructing the equations of moments $\langle U\rangle$, $\langle V\rangle$, $\langle U^2\rangle$, $\langle UV\rangle$, $\langle V^2\rangle$, $\langle U^3\rangle$,$\langle V^3\rangle$, $\langle U^2V\rangle$, $\langle UV^2\rangle$. These equations form a set of linear equations which can be put in the form

$$\dot{L} = AL \quad (38)$$

where L is a 9-component vector and A is a 9×9 matrix as given in the Appendix. Care must be taken to note that to close the set of equations we have discarded terms beyond fourth order but not taken resort to factorization of higher moments to lower ones. The procedure may be continued upto any nth order moment while truncating the hierarchy at $(n+1)$th level.

To proceed further we now consider the solution of the equation (38). Expressing the solution of equation (38) in the form $L(t)\sim e^{\lambda t}$ we have the following determinental equation for the eigenvalue problem,

$$\mid A - \lambda I \mid = 0 \quad (39)$$

where λ is the eigenvalue and I refers to a 9×9 unit matrix. We start with the condition $k=0$ in absence of any noise corresponding to a dynamical stable state. For $k\neq 0$ and in absence of noise the system may still remain in the homogeneous stable state provided one chooses the parameter space in such a way that the diffusion-driven (Turing) instability does not arise.

The condition for instability due to additive noise is that when at least one of the eigenvalues

$$\mathrm{Re}\lambda(k^2) > 0 \quad (40)$$

for the given set of parameters in the dynamics. This implies that the spatiotemporal perturbations $\langle\delta u(x,y,t)\rangle$ and $\langle\delta v(x,y,t)\rangle$ diverges in time as $e^{\lambda t}$. The structure of the 9×9 stability matrix is too complex and therefore precludes the possibility of determination of eigenvalues analytically. We therefore need to take resort to numerical simulation to find out the range of k^2 for which Reλ is positive. The existence of this range of wave numbers indicates that fluctuations in a certain wave number range which makes at least one of Reλ positive only exhibits noise-induced instability. Before switching over to these numerical aspects and the explicit examples we emphasize several pertinent points:

(i) A close look into the equations for moments (38) reveals that the spatiotemporal perturbation with cosine periodicity picks up the appropriate harmonic components from cubic nonlinearity of the reaction terms. Thus the first nontrivial and nonvanishing leading order contribution to average dynamics and instability in presence of additive noise is the cubic nonlinearity. For the sake of clarity and to capture the essential physics we have not considered the terms beyond third order in constructing the equations for moments. For a better numerical accuracy one may extend the calculations by incorporating the appropriate odd order terms, i.e. fifth order and beyond.

(ii) The distinction between the additive noise-induced instability as discussed here and the Turing instability is noteworthy. While Turing instability requires the diffusion coefficients of the two species to differ significantly and follows from a linear analysis, it is evident that the variation of the ratio of the diffusion coefficient is not restrictive in the case of additive noise induced instability which follows from the analysis of higher moments.

(iii) The analysis of first moments calculated with the help of Novikov's theorem directly, is sufficient in the case of reaction- diffusion system driven by multiplicative noise. On the other hand the higher moments are essential for capturing the threshold condition for instability due to additive noise. It is also worth- noting that the dispersion relations (Re$\lambda(k^2)$ vs. k^2) based on the two analysis are quite distinct.

(iv) An important content of the present analysis is that we use truncation at higher moments rather than factorization to close the set of equations. While truncation results in linear set of equations, factorization makes the system of equations nonlinear.

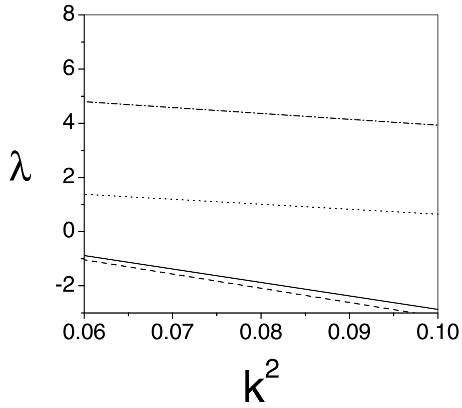

Fig. 1. Dispersion curves: Plot of the largest Reλ vs. k^2 for the CDIMA system for the parameter set $a = 18.0$, $b = 1.5$, $d = 1.6$, $\sigma = 6.0$, $D_v = 0.0$ at various noise strengths (a) $D_u = 0.0$ (dashed line) (b) $D_u = 0.001$ (solid line) (c) $D_u = 0.003$ (dotted line) (d) $D_u = 1.0$(dash-dotted line) (all the quantities are dimensionless).

3 Specific examples and numerical analysis

In order to explore the condition for instability induced by additive noise on a homogeneous steady state stabilized by diffusion we proceed by solving numerically the eigenvalue equation (38) to determine Re$\lambda(k^2)$ as a function of wavenumbers k^2 for a given parameter space. We consider two specific reaction-diffusion systems (i) CDIMA ($ClO_2 - I_2$-Malonic Acid) reaction and (ii) Pigmentation of fish model, to illustrate the basic issues considered in the last section.

3.1 Chlorine-dioxide-iodine-Malonic Acid (CDIMA) system:

This wellknown two-variable model proposed by Lengyel and Epstein is the reaction-diffusion system in which Turing pattern was first demonstrated experimentally [23,24]. The reaction-diffusion system is governed by the following equations:

$$u_t = a - u - 4au/(1+u^2) + u_{xx} + u_{yy} \tag{41}$$

$$v_t = \sigma[b(u - uv/(1+u^2)) + dv_{xx} + dv_{yy}]. \tag{42}$$

Here u and v correspond to the dimensionless concentrations of activator (ClO_2^-) and inhibitor (I^-), respectively. a and b are the parameters containing kinetic and thermodynamic constants as well the initial concentrations of the reacting species. d is the ratio of the diffusion coefficients $d(I^-)/d(ClO_2^-)$. σ is an experimentally adjustable parameter which can be varied by adjusting initial concentrations of starch, the complexing agent in the model and which plays an important role in determining the stability of various regimes. The unique steady state of the system is $u_0 = a/5$, $v_0 = 1 + a^2/25$, i.e., the homogeneous steady state is independent of b [25]. For other details we refer to Lengyel et al.

We fix the parameter space with $a = 18.0$, $b = 1.5$, and $d = 1.6$ and $\sigma = 6$. The system is homogeneously stable in absence of any external noise ($D_u = D_v = 0$) in this regime. This can be verified by noting that the largest eigenvalue (Reλ) determined by solving (38) numerically for the relevant range of wave numbers k^2 remains negative as shown in Figure 1. The effect of introduction of additive noise term $\eta(x, y, t)$ on u (no noise term is applied on v) on this homogeneous stable state is exhibited in Figure 1 for different noise strengths $D_u = 0.001$ and $D_u = 0.003$ and for $D_u = 1.0$. For small noise strength $D_u = 0.001$ the largest eigenvalue still remains negative rendering the system stable. Beyond a critical noise strength around $D_u = 0.003$, Reλ becomes positive and the system loses its stability for a range of wave numbers. The allowed number of modes increases further on increasing the noise strength to higher values.

We now proceed to analyze how far our theoretical analysis of onset of instability by additive noise corresponds to direct numerical simulation of equations (41) and (42) for CDIMA system driven by noise $\eta(t)$ only. The average and noise correlation of $\eta(t)$ are given by equations (5) and (6), respectively. The computations are performed by explicit Euler method on a two-dimensional grid 200 × 200 with $\triangle x = \triangle y = 0.25$ and time step $\triangle t = 0.0005$ and periodic boundary conditions. The Gaussian, white noise is generated with the help of Box-Muller algorithm. The simulations are started with a small fixed spatial perturbation of ∼1% around the steady state. The given parameter set in absence of any noise ($D_u = 0.0$) keeps the system in a homogeneous steady state. This is shown in Figure 2a. When a small noise of strength ($D_u = 0.001$) is applied on the system, the system exhibits patches of inhomogeniety, which, however, does not exhibit strict stationarity (Fig. 2b) in the long run. Around $D_u = 0.003$, one observes inhomogeneous spatial structure in the form of stationary spots. The pattern attains stationarity at around 10^5 time steps. This noise induced pattern is shown in Figure 2c. At higher noise strengths ($D_u > 1.0$) the pattern tends to loose

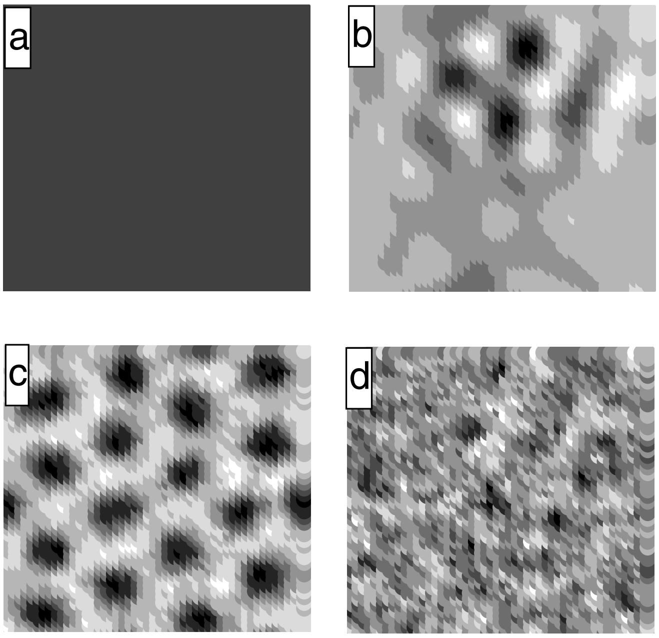

Fig. 2. Numerically simulated noise induced spatial pattern in CDIMA system for $a = 18.0$, $b = 1.5$, $\delta = 1.6$, $\sigma = 6.0$, $D_v = 0.0$. Bright pixels represent high I^- concentration (a) $D_u = 0.0$ (b) $D_u = 0.001$ (c) $D_u = 0.003$ (d) $D_u = 1.0$.

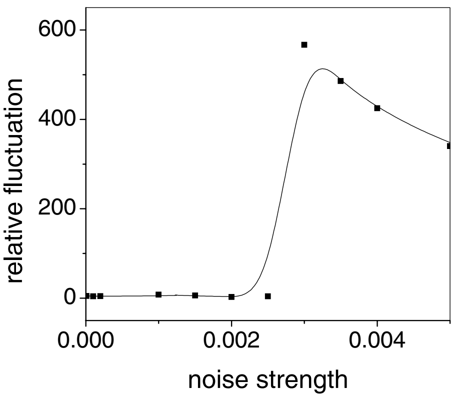

Fig. 3. The plot of relative fluctuation of order parameter against noise strength for CDIMA system in the parameter range given in Figures 1 and 2.

its coherent structure with considerable deformation as shown in Figure 2d.

In order to understand the correspondence between the theory and numerical simulation it is worthwhile to look into the behaviour of the relative fluctuation of an appropriate order parameter at the nonequilibrium phase transition point. To this end we introduce M, an intensive order parameter as $M =| \Sigma_{i=1,N} u_i |$, where u_i is the steady state concentration of variable u at a discrete site i (sum indicates the total contribution of all sites). Following reference [5] it is then possible to define a generalized susceptibility which is a measure of relative fluctuation of the order parameter $\chi = [\langle M^2 \rangle - \langle M \rangle^2]/N^2 D_u$. In Figure 3 we calculate numerically this relative fluctuation as a function of noise strength D_u for the same set of parameter values of a, b, σ. Simulations are performed from a homogeneous state of the system to an inhomogeneous state of pattern, each state being considered after the system reaches its stationarity. It is therefore easy to locate the nonequilibrium phase transition point corresponding to a critical value of noise strength $D_u = 0.003$ in Figure 3, and observe a fair correspondence between the theoretical and the numerical values. We have carried out further simulation on the system for the same set of experimentally admissible parameters for several other values of σ and found similar noise-induced transitions for $\sigma > 5.6$. The result remain qualitatively same which for the sake of brevity have not been reproduced here.

3.2 Pigmentation of fish model

As a second example we now consider another two variable reaction-diffusion system [26,27]. This model was proposed by Bario et al. as an alternative approach to mechanochemical models where pattern arises due to physical interaction between cells with external surrounding leading to cell aggregation and differentiation. The equations are given by

$$\frac{\partial u}{\partial t} = \alpha u(1 - r_1 v^2) + v(1 - r_2 u) + \delta \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \tag{43}$$

$$\frac{\partial v}{\partial t} = \beta v(1 + \alpha r_1 u v/\beta) + u(\gamma + r_2 v) + \delta d \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \tag{44}$$

where α, β, γ, r_1, r_2 are the given parameters of the dynamics. δ is the length scale. The choice of reaction terms is motivated by requirement of conservation of certain chemical species and nonlinearity which determines the specific unstable modes to dominate for the selection of a typical pattern when Turing instability sets in. Since in the absence of diffusion the system admits one more solution at $v = -(\alpha+\gamma)u/(1+\beta)$ which follows simply from homogeneous steady state conditions on (43) and (44), we ensure the steady state (0,0) (which is the homogeneous steady state of the system) as the only uniform steady state by setting the parameter $\alpha = -\gamma$. The complex pattern generated with this model under various conditions bear striking resemblance with pigmentation patterns observed in a number of fish species. For details we refer to Bario et al. [26].

The parameter set chosen for the present investigation of noise-induced instability is $\alpha = 0.899$, $\beta = -0.91$, $\gamma = -0.899$, $\delta = 2.0$, $r_1 = 0.02$, $r_2 = 0.2$, $d = 0.6$. This set admits of a homogeneous steady state in absence of any noise term. This may also be ascertained by solving equation (38) for Reλ when noise terms are switched off. We find that all the Reλ values are negative for an admissible range of wave numbers. A typical plot is shown in

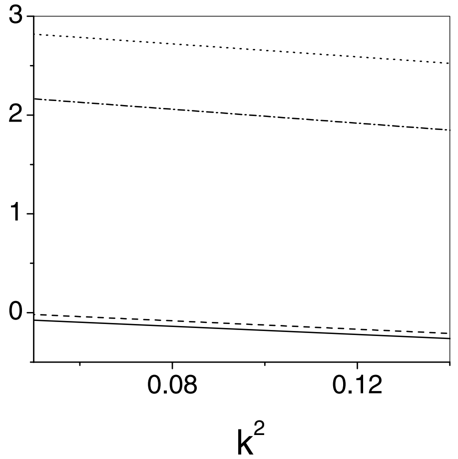

Fig. 4. Dispersion curves: Plot of the largest Reλ vs. k^2 for the pigmentation of fish model for the parameter set $\alpha = 0.899$, $\beta = -0.91$, $\gamma = -0.899$, $\delta = 2.0$, $r_1 = 0.02$, $r_2 = 0.2$, $d = 0.6$ at various noise strengths (a)$D_u = D_v = 0.0$ (bold line), (b) $D_u = D_v = 6.0\times10^{-4}$ (dashed line), (c) $D_u = D_v = 5.0\times10^{-2}$ (dash-dotted line), (d) $D_u = D_v = 0.1$ (dotted line) (all the quantities are dimensionless).

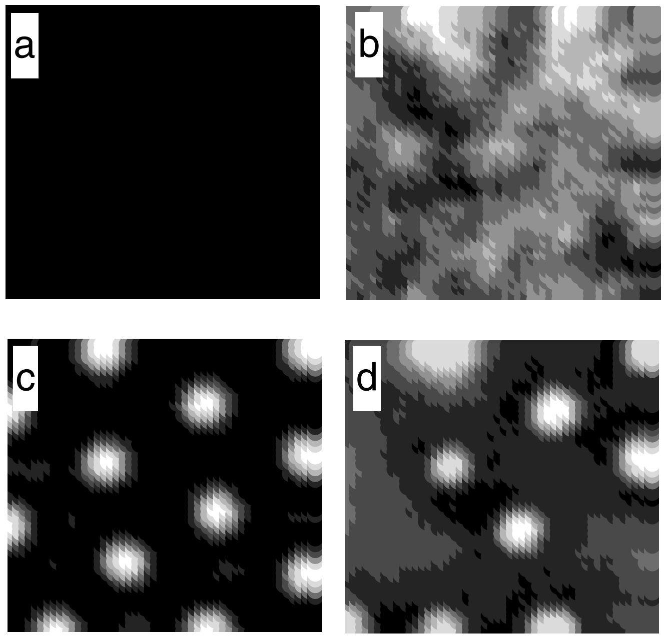

Fig. 5. Numerically simulated noise induced spatial pattern for the pigmentation of fish model for the parameter set $\alpha = 0.899$, $\beta = -0.91$, $\gamma = -0.899$, $\delta = 2.0$, $r_1 = 0.02$, $r_2 = 0.2$, $d = 0.6$, at various noise strengths (a) $D_u = D_v = 0.00$, (b) $D_u = D_v = 6.0 \times 10^{-4}$, (c) $D_u = D_v = 0.05$, (d) $D_u = D_v = 0.1$ (all the quantities are dimensionless).

Figure 4 by the solid line. The effect of introduction of additive noise terms $\eta(x, y, t)$ and $\xi(x, y, t)$ on (43) and (44) can now be realized through nonzero D_u and D_v terms. As the noise strength ($D_u = D_v$) is varied it has been observed that above a critical value of the noise strength around $D_u = D_v = 0.002$ the largest Reλ becomes positive for a finite range of k^2 values. The typical dispersion relations are shown in Figure 4 for $D^u = D^v = 6 \times 10^{-5}$, $D_u = D_v = 5 \times 10^{-2}$, $D_u = D_v = 0.1$. For higher noise strengths a wider range of nodes with positive eigenvalues are admitted.

In order to corroborate the above theoretical analysis we carry out a direct numerical simulation of equations (43) and (44) with additive noise terms $\eta(x, y, t)$ and $\xi(x, y, t)$. The computations are performed on a two dimensional grid 200 × 200 with $\triangle x = \triangle y = 1.0$ and time step $\triangle t = 0.05$ and zero flux boundary conditions. The reaction-diffusion system in this parameter regime admits of homogeneous steady state in absence of noise as shown in Figure 5a. We observe that as the noise strength $D_u(= D_v)$ is gradually increased instability sets in and an inhomogeneous spatial structure appears beyond a critical value of the noise strength as predicted by instability analysis. However this inhomogeneous structure does not attain stationarity in the long time limit. Nonstationary inhomogeneous structure persists even at somewhat higher noise strengths. Stationarity however is attained at around $D_u = D_v = 5 \times 10^{-2}$. The typical patterns are shown in Figure 5 for $D_u = D_v = 6.0 \times 10^{-5}$, $D_u = D_v = 5 \times 10^{-2}$ and $D_u = D_v = 0.1$. The pattern in Figure 5c attains stationarity at around 2×10^4 time steps and tends to get deformed and loose its coherent structure as the noise strength is increased to a relatively higher value. It is important to note that our noise-induced stability analysis can predict initiation of pattern but does not ensure its stationarity.

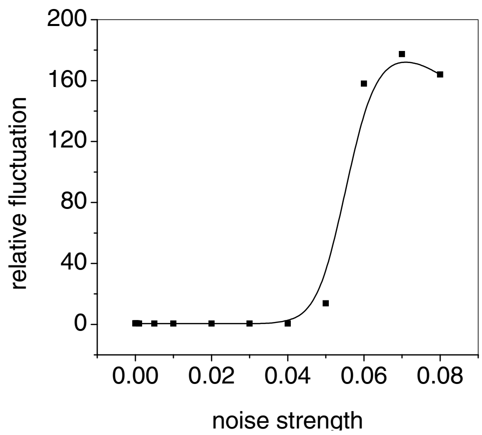

Fig. 6. The plot of relative fluctuation of order parameter against noise strength for pigmentation of fish model in the parameter range given in Figures 4 and 5.

In order to locate the nonequilibrium transition point we plot in Figure 6 the relative fluctuation of the order parameter as a function of noise strength for the same parameter set used in Figure 5. It is evident that at around $D_u = 0.05$ the profile depicts a sharp transition point which corresponds fairly well to Figure 5c and positivity of the largest eigenvalues of the stability matrix which allows a number of modes to be unstable over a range

$$A = \begin{pmatrix} f_u - k^2 & f_v & 0 & 0 & 0 & 3/32.f_{uuu} & 3/32.f_{vvv} & 9/32.f_{uuv} & 9/32.f_{uvv} \\ g_u & g_v - dk^2 & 0 & 0 & 0 & 3/32.g_{uuu} & 3/32.g_{vvv} & 9/32.g_{uuv} & 9/32.g_{uvv} \\ 2C^u & 0 & 2(f_u - k^2) & 0 & 2f_v & 0 & 0 & 0 & 0 \\ 0 & 2C^v & 0 & 2g_v - 2dk^2 & 2g_u & 0 & 0 & 0 & 0 \\ C^v & C^u & g_u & f_v & \begin{matrix} f_u + g_v \\ -(1+d)k^2 \end{matrix} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \begin{matrix} 3f_u - 3k^2 \\ +6C^u \end{matrix} & 0 & 3f_v & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \begin{matrix} 3g_v - \\ 3dk^2 + 6C^v \end{matrix} & 0 & 3g_u \\ 0 & 0 & 0 & 0 & 0 & g_u + 2C^v & 0 & \begin{matrix} g_v - dk^2 - 2k^2 \\ +2f_u + 2C^v \end{matrix} & 2f_v + 2C^u \\ 0 & 0 & 0 & 0 & 0 & 0 & f_v + 2C^u & 2g_u + 2C^u & \begin{matrix} f_u - k^2 - \\ 2dk^2 + 2C^v \end{matrix} \end{pmatrix}$$

$$L = \begin{pmatrix} w_u \\ w_v \\ w_{u^2} \\ w_{v^2} \\ w_{uv} \\ w_{u^3} \\ w_{v^3} \\ w_{u^2v} \\ w_{uv^2} \end{pmatrix}$$

of wavelengths. We mention in passing that one observes similar correspondence for other parameter values, the results remain qualitatively same.

4 Conclusion

Additive noise can induce a stationary spatial pattern on a homogeneous state stabilized by diffusion in a reaction-diffusion system. We have made a theoretical analysis of the problem to determine the instability condition for pattern formation and performed numerical simulations on two specific reaction-diffusion systems to corroborate the analysis. We summarize the main conclusions of this study as follows:

(i) Our theoretical analysis of the condition for instability induced by additive noise is based on a system of linear equations of higher moments. An important advantage of the scheme is that although we make truncation at a higher moment to close the hierarchy of equations this is free from any approximation of factorization of higher moments into lower ones. This allows us to take care of the correlations and the interplay of nonlinearity and stochasticity in the dynamics in an useful way.

(ii) We have shown that the third order or cubic nonlinearity plays a crucial role in determining the instability threshold for the additive noise to a leading order. A close look into present treatment reveals that for multiplicative noise, a quadratic nonlinearity and a direct implementation of Novikov's theorem for calculation of average of the variables with noise in an analysis of linear moments may lead to instability resulting in inhomogeneous spatial structure.

The method followed in the paper is simple and is equipped to deal with multiplicative noise as well. It can be applied to reaction-diffusion systems for several variables and extended to keep track of the higher order nonlinearity in pattern selection and formation. We hope that the procedure can also be applied to several other processes in spatially extended systems induced by additive noise.

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Appendix

see equation above

References

1. W. Horsthemke, M. Malek Mansour, Z Phys. B **24**, 30 (1976)
2. W. Horsthemke, R. Lefever, *Noise-induced Transitions* (Springer, Berlin, 1984)
3. R. Benzi, A. Sutera, A. Vulpiani, J. Phys. A **14**, L453 (1981)
4. L. Gammaitoni, P. Hanggi, P. Jung, F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998)
5. J. Garcia-Ojalvo, J.M. Sancho, *Noise in Spatially Extended Systems* (Springer, New York, 1999)
6. R. Müler, K. Lippert, A. Kuhnel, U. Behn, Phys. Rev. E **56**, 2658 (1997)

7. O. Carrillo, M. Ibanes, J.M. Sancho, Fluct. Noise Lett. **2**, LI (2002)
8. C. Van den Broeck, J.M.R. Parrondo, R. Toral, Phys. Rev. Lett. **73** 3395 (1994)
9. C. Van den Broeck, J. M. R. Parrondo, R. Toral and R. Kawai, Phys. Rev. E **55**, 4084 (1997)
10. J. Garcia-Ojalvo, J.M.R. Parrondo, J.M. Sancho, C. Van den Broeck, Phys. Rev. E **54**, 6918 (1996)
11. M.G. Zimmermann, R. Toral, O. Piro, M. San Miguel Phys. Rev. Lett. **85**, 3612 (2000)
12. L. Q. Zhou, X. Jia, Q. Ouyang, Phys. Rev. Lett. **88**, 138301 (2002)
13. J. Garcia-Ojalvo, A. Hernadez-Machado, J.M. Sancho, Phys. Rev. Lett. **71**, 1542 (1993)
14. A. Becker, L. Kramer, Phys. Rev. Lett. **73**, 955 (1994); A. Becker, L. Kramer, Physica D **90**, 408 (1995)
15. R. Kawai, X. Sailer, L. Schimansky-Geier, C. Van den Broeck, Phys. Rev. E **69**, 051104 (2004)
16. J.M.R. Parrondo, C. Van den Broeck, J. Buceta, F.J. de la Rubia, Physica A **224**, 153 (1996)
17. A. A. Zaikin, L. Schimansky-Geier, Phys. Rev. E. **58**, 4355 (1998)
18. J. Buceta, M. Ibanes, J.M. Sancho, K. Lindenberg, Phys. Rev. E **67**, 021113 (2003)
19. J.B. Swift, P.C. Hohenberg, Phys. Rev. A **15**, 319 (1977)
20. F. Lesmes, D. Hochberg, F. Moran, J. Perez-Mercader, Phys. Rev. Lett. **91**, 238301-1 (2003)
21. G. Jumarie, *J. Franklin Inst. B*, **332**, 761 (1995)
22. L.S. Socha, T.T. Soony, Appl. Mech. Rev. **44**, 399 (1991)
23. I. Lengyel, Gy.Rabai, I.R. Epstein, J. Am. Chem. Soc. **112**, 4606 (1990)
24. V. Castets, E. Dulos, J. Boissonade, P. De Keeper, Phys. Rev. Lett. **64**, 2953 (1990)
25. S. Kar, J.K. Bhattacharjee, D.S. Ray, Eur. Phys. J. B **43**, 109 (2005)
26. R.A. Bario, C. Varea, J. Aragon, P.K. Maini, Bull. Math. Biol. **61**, 483 (1999)
27. S.S. Riaz, S. Kar, D.S. Ray, J. Chem. Phys. **121**, 5395 (2004)